Generalized evolution of bivariate copulas in discrete processes

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Abstract: There has been much interest in copulas, which are known to provide a flexible tool for analyzing the dependence structure among random variables. Dependence relations must be dynamic rather than static in nature. However, copulas are useful mainly for static matters; their definitions themselves do not contain time variables. Thus we introduce evolving copulas, which transform through time autonomously as governed by recurrence relation.

Keywords: Quantitative Risk Management; Copula; Recurrence relation

1. Introduction

Dependence relations among random variables are one of the most important subjects for probability and statistics study. Analyzing dependence structures is crucial from both theoretical and applied viewpoints. Recently, members of the financial sector, like insurance companies and their regulators, have recognized that it is critical to manage these risks in a sophisticated way, that is, quantitatively. Quantitatively measured risks play a central role in this management framework. These entities face many kinds of risks, and the relations between them are very complicated. Thus, it is important to reflect the dependence relations to measure the risks quantitatively: the more dependence there is among risks, the less aggregated the risks are. See Yoshizawa, Y. [19], [20].

Linear correlation is often recognized as a satisfactory measure of dependence in risk management. However, it cannot capture the non-linear dependence relations that exist among many risk factors. See Embrechits, P. et al. [3]. What expresses the dependence relations among risk factors? If we capture a multivariate joint distribution of all the risk factors, we can recognize their dependence structure probabilistically or statistically.

For this reason, there has been much interest in a copula. A copula links multivariate joint distribution and univariate marginal distributions. Copulas are often employed to investigate the dependence structure among random variables. The fundamental theorem of copulas, "Sklar's theorem," first appeared in 1959. See Schweizer, B., & Sklar, A. [15], and Sklar, A. [16]. This elegant theorem claims that all multivariate distributions have copulas, and that copulas are used in conjunction with univariate distributions to construct multivariate distribution. As for the study of copulas, Free, E.W. et al. [4] and Tsukahara, H. [18] provide useful introductive reviews. Nelsen, R.B. [14] wrote the excellent standard textbook, providing a systematic development of the theory of copulas, particularly bivariate copulas. Moreover, McNeil, A. J. et al. [12] introduce copulas theoretically and practically, examining quantitative risk management for financial sectors. Copulas have been extensively studied and applied in a wide range of areas concerning dependence relations, because of their flexibility. See Breymann, W. et al. [1], Genest, G. et al. [5], and Goorbergh, R.W.J. et al. [6].We summarize the basic concept of copulas and rank correlations in Section 2.

Copulas are useful mainly for static matters; their definitions themselves do not contain time variables. Mikosch, T. [13] suggests, "Copulas do not fit into the existing framework of stochastic processes and time series analysis; they are essentially static models and are not useful for modeling dependence through time". However a few exceptions exist: copulas and the markov process, as in Darsow, W.F. et al. [2], and dynamic copulas, as in Patton, A.J. [17]. In these articles copulas and Markov processes can be used to analyze the dependence relations between markov processes at different times, and dynamic copulas involve the development of dynamic time series models for financial return data using conditional copulas.

It is well known that rank correlations, one of the prevailing measures of dependence, are derived only by copulas. That is to say, copulas determine rank correlations. Therefore, it is natural to analyze only copulas in the study of transformations of dependence structures through time. As a first step, we start to investigate how copulas transform, and if they evolve in accordance with the heat equation, which is one of the basic partial differential equations used to describe dynamic movements. We already studied the evolution of bivariate copulas in continuous and discrete processes, and summarize them in Section 3.

However the evolution of copulas is not versatile; it is restricted to events whose essential dependences do not fluctuate but transform monotonically. The evolution of copulas fits with events where dependence decreases. In this paper, we extend the evolution of bivariate copulas in discrete processes to the generalized evolution of copulas, which is governed by past copulas asymmetrically weighted by coefficients. We propose a prototype of the generalized evolution of bivariate copulas in Section 4, and comment on their expected collaboration with Artificial Intelligence (AI) in Section 5. We hope the generalized evolution of copulas fits into various transformations of dependence structures, such as rapidly strengthening dependences, becoming independent smoothly and so on.

2. Copulas and rank correlation

In this section, as an introduction, we summarize basic concept of copulas and rank correlations.

2.1 Basic concept of copulas

We marshal the basic concepts and properties of bivariate copulas for the preparation of the subsequent sections. We illustrate the definition of bivariate copulas; describe Sklar's theorem, which plays a central role in copula theories.

The bivariate copulas are defined by some conditions described in the following (1) and (2). The property (1) is called 2-increasing condition, and the properties (2) are boundary conditions.

Copula. Bivariate copula is a function $C(u,v)$ from I^2 to I, which is defined by the following conditions

2-increasing condition;

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0, \quad \text{for } (u_i,v_j) \in I^2 \ (i,j=1,2), u_1 \leq u_2, v_1 \leq v_2. \tag{1}$$

Boundary conditions;

$$C(u,0) = C(0,v) = 0, \quad C(u,1) = u \text{ and } C(1,v) = v. \tag{2}$$

The following Sklar's theorem is the core theory among various copula theories. Thanks to this theorem we can construct multivariate distribution by coupling univariate marginal distributions.

Sklar's theorem. Let *H* be a bivariate joint distribution function with marginal distribution function *F* and *G*; that is

$$\lim_{y\to\infty} H(x,y) = F(x) = u \quad \text{and} \quad \lim_{x\to\infty} H(x,y) = G(y) = v$$

Then there exists a copula, which is uniquely determined on Ran *F* × *Ran G* such that

$$H(x,y) = C\left(F(x), G(y)\right) = C(u,v). \tag{3}$$

Conversely, if *C* is a copula and *F* and *G* are distribution functions, then the function *H* defined by (3) is a bivariate joint distribution function with marginal distribution functions *F* and *G*.

There are copulas for sample data, which are called empirical copulas. Empirical copulas are used for evolution of copulas in discrete processes. We explain empirical copulas, empirical copula frequency functions and the relation between them with reference to Nelson, R.B. [14].

Empirical copulas. Let $\{x_k, y_k\}_{k=1}^{n}$ denote a sample with size n from a continuous bivariate distribution. The empirical copulas C_n is given by

$$C_n\left(\frac{i}{n},\frac{j}{n}\right) = \frac{\text{number of pairs } (x,y) \text{in sample with } x \leq x_{(i)}, y \leq y_{(j)}}{n^2}, \tag{4}$$

The empirical copula frequency function c_n is given by

$$c_n\left(\frac{i}{n},\frac{j}{n}\right)=\begin{cases}\frac{1}{n}, & if\ \left(x_{(i)},y_{(j)}\right) \text{ is an element of the sample,}\\ 0, & \text{otherwise}\end{cases}$$

where $x_{(i)}$ and $y_{(j)}$, $1 \le i,j \le n$, denote order statistics from the sample.

Furthermore the relation between C_n and c_n is deduced by the definition as

$$C_n\left(\frac{i}{n},\frac{j}{n}\right)=\sum_{p=1}^{i}\sum_{q=1}^{j} c_n\left(\frac{p}{n},\frac{q}{n}\right), \text{ and} \tag{5}$$

$$c_n\left(\frac{i}{n},\frac{j}{n}\right)=C_n\left(\frac{i}{n},\frac{j}{n}\right)-C_n\left(\frac{i-1}{n},\frac{j}{n}\right)-C_n\left(\frac{i}{n},\frac{j-1}{n}\right)+C_n\left(\frac{i-1}{n},\frac{j-1}{n}\right).$$

2.2 Basic concept of rank correlations

Rank correlations are a sort of dependence measures for two variables, and their special feature is that they do not depend on marginal distributions, but depend only on their copulas. That is to say, copulas determine rank correlations

We introduce Kendall's tau (τ) and Spearman's rho (ρ), which are typical rank correlations. Their definitions, their popular version derived by copulas and their discrete versions are as follows.

Kendall's tau. Kendall's tau for a pair (X,Y) is the probability of concordance minus the probability of dis- concordance. Kendall's tau is defined as

$$\tau_{X,Y} \coloneqq P[(X_1-X_2)(Y_1-Y_2)>0]-P[(X_1-X_2)(Y_1-Y_2)<0],$$

where (X_1, Y_1) and (X_2, Y_2) are independent and identically distributed random vector with joint distribution.

Moreover the popular version of Kendall's tau is derived using only copula as

$$\tau_{X,Y}=4\iint_{I^2} C(u,v)dC(u,v)-1, \tag{6}$$

where u and v are uniform random variables, and their discrete version are derived by solely empirical copula as

$$\frac{2n}{n-1}\sum_{i=2}^{n}\sum_{j=2}^{n}\sum_{p=1}^{i-1}\sum_{q=1}^{j-1}\left[c_n\left(\frac{i}{n},\frac{j}{n}\right)c_n\left(\frac{p}{n},\frac{q}{n}\right)-c_n\left(\frac{i}{n},\frac{q}{n}\right)c_n\left(\frac{p}{n},\frac{j}{n}\right)\right] \tag{7}$$

$$=\frac{2n}{n-1}\sum_{i=2}^{n}\sum_{j=2}^{n}\left[C_n\left(\frac{i}{n},\frac{j}{n}\right)C_n\left(\frac{i-1}{n},\frac{j-1}{n}\right)-C_n\left(\frac{i}{n},\frac{j-1}{n}\right)C_n\left(\frac{i-1}{n},\frac{j}{n}\right)\right].$$

Spearman's rho. Spearman's rho for (X_1,Y_1) and (X_2,Y_3) is proportional to the probability of concordance minus the probability of dis-cordance. Spearman's rho is defined as

$$\rho_{X,Y} := 3(P[(X_1-X_2)(Y_1-Y_3)>0]-P[(X_1-X_2)(Y_1-Y_3)<0]),$$

where (X_1,Y_1), (X_2,Y_2) and (X_3,Y_3) are independent and identically distributed random vectors with a common joint distribution functions.

The well-known version of Spearman's rho is derived only by copula as

$$\rho_{X,Y}=12\iint_{I^2} uv\,dC(u,v)-3=12\iint_{I^2} C(u,v)dudv-3, \tag{8}$$

where u and v are uniform random variables, where u and v are uniform random variables, and their discrete version is derived by solely empirical copula as

$$\frac{12}{n^2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[C_n\left(\frac{i}{n},\frac{j}{n}\right)-\frac{i}{n}\cdot\frac{j}{n}\right]. \tag{9}$$

3. Evolution of copulas

In this section, we summarize continuous evolution of bivariate copulas and discrete evolution of bivariate copulas. The reason we adopt the heat equation to evolve copulas is as follows.

a) Rank correlations, one of the prevailing measures of dependence, are derived only by copulas. That is to say, copulas determine rank correlations. Thus it is natural to analyze only copulas in the study of transformations of dependence structures through time.

b) The object of our study is autonomous movements of dependencies. The heat equation is one of the basic partial differential equations used to describe dynamic autonomous movements. Therefore we adopt it as a first step.

3.1 Evolution of copulas in continuous processes

We proved the existences and solutions of evolution of copulas in continuous processes, and their convergence to the product copula. Moreover we proved that rank correlations of evolution of copulas converge to zero exponentially as $t \to \infty$. See Yoshizawa,Y.[23], Yoshizawa, Y., & Ishimura, N. [21] and Ishimura, N., & Yoshizawa, Y. [7],[8].

Continuous evolution of copulas. For any bivariate copula $C_0(u,v)$, there exists a unique family of time dependent bivariate copula $\{C(u,v,t)\}_{t\geq 0}$, which satisfies the heat equation

$$\frac{\partial C}{\partial t}(u,v,t) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) c(u,v,t), \tag{10}$$

for $(u,v,t) \in I^2 \times (0,\infty), where\ C(u,v,t) = C_0(u,v) \quad on\ (u,v) \in I^2$.

The solution of the partial differential equation (10) is

$$C(u,v,t) = uv + 4\sum_{m,n=1}^{\infty} e^{-\pi^2(m^2+n^2)t} \sin m\pi u \sin n\pi v\, K(m,n)\,, \tag{11}$$

where $K(m,n) = \iint_{I^2} \sin m\pi\xi \sin n\pi\eta \ \ (C_0(\xi,\eta) - \xi\eta) d\xi\, d\eta$.

Convergence to the product copula. Evolution of bivariate copulas converge to the product copula, which means independences between variables, as

$$\lim_{t\to\infty} C(u,v,t) = \Pi(u,v) \quad \text{uniformly on } (u,v) \in I^2, \quad \text{where} \quad \Pi(u,v) := uv. \tag{12}$$

Rank correlation. Rank correlations are a sort of dependence measures, and Kendall’s tau and Spearman’s rho are typical rank correlations. Their special feature is that they do not depend on marginal distributions, but depend only on their copulas. Furthermore rank correlations, Kendall’s tau (τ_{C_t}) and Spearman’ rho (ρ_{C_t}), of evolution of copulas $C(u,v,t)$, converge to zero exponentially as $t \to \infty$.

3.2 Evolution of copulas in discrete processes

In general, it is difficult to solve partial differential equations analytically; therefore, numerical approaches are often applied in practice, especially where analytical solutions do not exist. Moreover, numerical analysis is suitable for computer calculation.

Therefore we studied and created the following discrete evolution of bivariate copulas which satisfy the following discrete version of the heat equation. See Yoshizawa,Y.[23] and Yoshizawa, Y., & Ishimura, N.[22].

Let $N \geq 0$ and $0 < h \leq 1$, we put $\triangle u = \triangle v := \frac{1}{N} = M,\ \triangle t := h,\ \lambda := \frac{\triangle t}{(\triangle u)^2} = \frac{\triangle t}{(\triangle v)^2} = hN^2$, and

$u_i := i \triangle u = \frac{i}{N}, v_j := j \triangle v = \frac{j}{N}$, for $i,j = 0,1,\cdots,N$.

At any $\{(u_i,v_j)\}_{i,j=0,1,2,\cdots,N}$, the value $C^n_{i,j} := C^n(u_i,v_i)$ is governed by the system of the difference equation

$$\frac{C^{n+1}_{i,j} - C^n_{i,j}}{\triangle t} = \frac{C^n_{i+1,j} - 2C^n_{i,j} + C^n_{i-1,j}}{(\triangle u)^2} + \frac{C^n_{i,j+1} - 2C^n_{i,j} + C^n_{i,j-1}}{(\triangle v)^2}, \qquad \text{for } i,j = 0,1,\cdots,N-1,$$

where $C_{i,j}^{0} = C^{0}(u_i, v_j) := C_0(u_i, v_j)$ is an initial copula, together with the boundary conditions as $C_{i,0}^{n} = C_{0,j}^{n} = 0$, $C_{i,N}^{n} = u_i, C_{N,j}^{n} = v_j$, for $i, j = 0,1, \cdots, N$, and the constrain $0 \leq \lambda \leq \frac{1}{4}$. The above deference equation is rewritten as

$$C_{i,j}^{n+1} = (1-4\lambda)C_{i,j}^{n} + \lambda\left(C_{i+1,j}^{n} + C_{i-1,j}^{n} + C_{i,j+1}^{n} + C_{i,j-1}^{n}\right) \tag{13}$$

The image of the deference equation (13) is shown in Figure 1.

Furthermore we define the interpolation as

$$C^{n}(u,v) := C_{i,j}^{n} + \frac{C_{i+1,j}^{n} - C_{i,j}^{n}}{u_{i+1}-u_i}(u-u_i) + \frac{C_{i,j+1}^{n} - C_{i,j}^{n}}{v_{j+1}-v_j}(v-v_j) \quad + \frac{C_{i+1,j+1}^{n} - C_{i+1,j}^{n} - C_{i,j+1}^{n} + C_{i,j}^{n}}{(u_{i+1}-u_i)(v_{j+1}-v_j)}(u-u_i)(v-v_j), \tag{14}$$

for $u_i \leq u \leq u_{i+1}, v_j \leq v \leq v_{j+1}$ and $(u, v) \in I^2$.

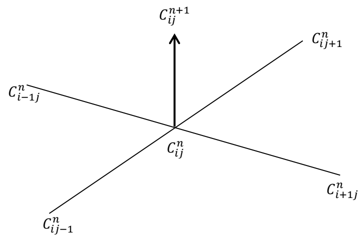


Figure 1. Imagae of deference equation

Discrete evolution of bivariate copulas. For any initial copula C_0 , there exists a sequence of copula $\{C^{n}(u,v)\}_{n=0,1,2,\cdots}$ on $(u, v) \in I^2$,which satisfy the system of difference equation at every $\{(u_i, v_j)\}_{i,j=0,1,2,\cdots,N}$ and the above interpolation. We can call these copulas $C^{n}(u, v)$ as evolution of bivariate copulas in discrete processes, which satisfy 2-increasing condition (1) and boundary conditions (2).

Convergence to the product copula. We proved that discrete evolution of copulas $C^{n}(u, v)$ converge to the product copula $\Pi(u, v) = uv$ uniformly on I^2 as well as continuous evolution of copulas. It is remarkable that the convergences do not depend on the fineness of mesh M, but depend only on the number of times n.

Rank correlation. For any initial copulas C_0 and a sequence of discrete evolution of copulas $\{C^{n}(u,v)\}_{n=1,2,\cdots}$, Kendall's tau (τ_n) and Spearman' rho (ρ_n) converge to zero exponentially as $n \to \infty$, corresponding to the continuous type.

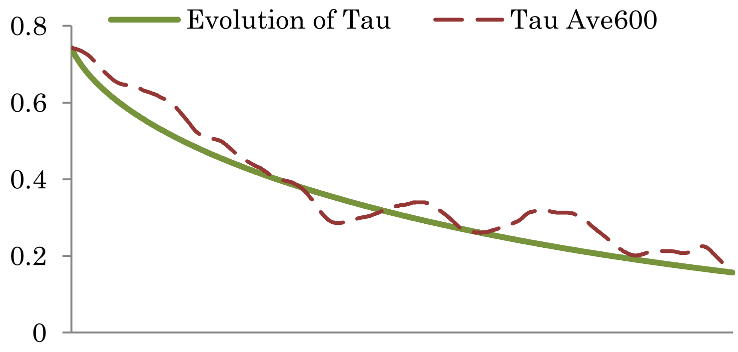
Convergence from discrete to continuous. We proved that discrete evolution copulas converge to continuous evolution of copulas as $N \to \infty$ and $h \to 0$ uniformly on $(u, v) \times t \in I^2 \times (0, \infty)$. Thus we can treat the discrete evolution of copulas as an approximation of the continuous evolution of copulas.

3.3 Application to empirical data

The evolution of copulas has properties to suit events whose dependence monotonically moving. Therefore, we focus on rapidly changing events when their directivities are almost stable. We select foreign exchange rates on January 15, 2015, when the Swiss franc endured a shock breakout after the announcement that the Swiss central bank had stopped monetary policy efforts to maintain the Swiss franc against the euro at more than 1.20. Therefore we analyze the dependence of euro–Japanese yen foreign exchange rates with those of the Swiss franc–Japanese yen.

First we collect data on the second time scale in order to capture their monotonic directivity. We construct empirical copulas of the euro–Japanese yen rates and the Swiss franc–Japanese yen rates for every second of 40 minutes using the formula (4). Then, we calculate their Kendall's tau correlation measure by using the formula (7). We apply a smoothing technique, a moving average method, to the transitions of Kendall's tau, since they fluctuate and include some singular data.

Next we compare Kendall's tau of the evolution of empirical copulas to the abovementioned moving averages of Kendall's tau of empirical copulas. We choose the start time when Kendall's tau of the empirical copula and its moving average are almost equal. We evolve the empirical copula at the start time using the deference equation (13) for 20 minutes. For reference, we plot the results again in Figure 2, which shows that the evolution of empirical copulas approximate the smoothed transition of empirical copulas from the viewpoint of Kendall's tau. See Ishimura, N., & Yoshizawa, Y. [11] and Yoshizawa,Y. [23] for the details.



Notes: TauAve600: Moving averages of 600 datasets of Kendall's tau for every second.

Evolution of Tau: Kendall's tau of evolution of copulas. We extract the empirical copula at around 18:30 when the Kendall's tau of the empirical copulas is almost equal to the smoothed Kendall's tau. We evolve them 1,200 times, which means for 20 minutes. For this evolution, we use the deference equation (13) with $\lambda = \frac{1}{5}$.

Data source: Bloomberg, exchange rate.

Figure 2. Kendall's tau of evolution of copulas

Using these flexible discrete copula models, we analyze the movement of dependence relations between the Swiss franc and the euro against the Japanese yen approximately. We can verify the practicality of some theories of the evolution of copulas, although this is restricted to events whose essential dependence does not fluctuate but transforms monotonically.

4. Generalized evolution of bivariate copulas in discrete processes

We are then able to generalize the difference equation (13) so that the asymmetrical weight is allowed as

$$C_{i,j}^{n+1} = \alpha C_{i,j}^{n} + \left(\beta_1 C_{i+1,j}^{n} + \beta_2 C_{i-1,j}^{n} + \beta_3 C_{i,j+1}^{n} + \beta_4 C_{i,j-1}^{n}\right), \tag{15}$$

where $\alpha > 0, \beta_k > 0\ (k = 1,2,3,4)$, and $\alpha + \sum_{k=1}^{4} \beta_k = 1$.
As for the detail, see Ishimura, N., & Yoshizawa, Y. [9].

Moreover we generalize this concept to propose the recurrence relation as

$$C_{i,j}^{n+1} := \sum_{k=-L}^{L} \sum_{l=-L}^{L} \alpha_{k,l}^{n} C_{i+k,j+l}^{n}, \tag{16}$$

for any n,
$\alpha_{k,l}^{n} \geq 0$ and $\sum_{k=-L}^{L} \sum_{l=-L}^{L} \alpha_{k,l}^{n} = 1$ for any $L \leq N$, and
$C_{i,j}^{n} = 0$ for $i \leq 0$ *or* $j \leq 0$, and $C_{i,j}^{n} = u_i$ for $j \geq N$ and $C_{i,j}^{n} = v_j$ for $i \geq N$.

First, we propose the recurrence relation (16) succeeds the conditions of copulas, and prove it in the following proposition 1.

Proposition 1. If $C_{i,j}^n$ satisfies the conditions of copulas (1) and (2), then $C_{i,j}^{n+1}$ defined by the recurrence relation (16) also satisfies the conditions of copulas.

(Proof) $C_{i+k,j+l}^n$ satisfies 2-increasing condition and $\alpha_{k,l}^n \geq 0$, thus $C_{i,j}^{n+1}$ also satisfies 2-incresing condition. $C_{i,j}^{n+1}$satisfies the boundary conditions according to its definition (16). Hence $C_{i,j}^{n+1}$ satisfies the conditions of copulas (1) and (2).

■

Secondly, we introduce the generalized evolution of bivariate copulas in discrete processes, generated by the recurrence relation and the interpolation , and prove that they are copulas in the theorem 2.

Theorem 2. (Generalized evolution of bivariate copulas in discrete processes). For any initial copula C_0, there exists a sequence of copulas $\{C^n(u,v)\}_{n=0,1,2,\cdots}$ on $(u,v) \in I^2$,which satisfy the system of the recurrence relation (16) at every $\{(u_i,v_j)\}_{i,j=0,1,2,\cdots,N}$ and the interpolation (16). We can call these copulas $C^n(u,v)$ as the generalized evolution of bivariate copulas in discrete processes, which satisfy 2-increasing condition (1) and boundary conditions (2).

(Proof) $C(u,v)$ is a function from I^2 to I, because $\alpha_{k,l}^n \geq 0$ and $\sum_{k=-L}^{L}\sum_{l=-L}^{L}\alpha_{k,l}^n = 1$ for any L ≤ N.
$C_{i,j}^{n+1}$satisfies the boundary conditions by the proposition 1, thus $C(u,v)$ also satisfies the boundary condition.
We verify 2-increasing condition. Let $u_i \leq u_1 \leq u_2 \leq u_{i+1}, v_j \leq v_1 \leq v_2 \leq v_{j+1}$, then using the proposition 1
$C^n(u_2,v_2) - C^n(u_2,v_1) - C^n(u_1,v_2) + C^n(u_1,v_1) = \left(C_{i+1,j+1}^n - C_{i+1,j}^n - C_{i,j+1}^n + C_{i,j}^n\right) \frac{(u_2-u_1)(v_2-v_1)}{(u_{i+1}-u_i)(v_{j+1}-v_j)} \geq 0.$
Hence we prove that $\{C^n(u,v)\}_{n=0,1,2,\cdots}$ are copulas.

■

Finally, as an example we show a generalized evolution of bivariate copula in discrete processes, which strengthen dependence relations through time, in the following example 3. For simplicity, we omit the interpolation (14).

Example 3. Let N=100, L=2, we set $\alpha_{k,l}^n$ as $\alpha_{2,2}^n = 1$, $\alpha_{k,l}^n = 0$ for $k \neq 2$ and $l \neq 2$. According to 2-increasing properties of $C_{i,j}^n$, the following equation holds as

$$C_{i,j}^{n+1} = \max_{-L \leq k \leq L, -l \leq l \leq L} C_{i+k,j+l}^n .$$

Thus we confirm that this generalized evolution of copula increase their dependences as time passing.

5. Discussion

In this paper, we extend our previous work, the evolution of bivariate copulas, to the generalized evolution of copulas in discrete processes. The traditional evolution of bivariate copulas have the properties to decrease dependence monotonically over time, therefore they fit with events whose dependence decrease. Applying the above generalized evolution of copula, we can manage the events whose dependences fluctuate. According to the mathematical equation (16) with the ideal coefficient $\alpha_{k,l}^n$, we can construct the dependence structure model transforming autonomously through time. The difficulty is to assume the proper coefficients $\alpha_{k,l}^n$ for any n, k, l.

Recently theory, technique and computer power for Artificial Intelligence (AI) are increasing their capacities, and AI becomes to be used widely. We think that AI, such as machine learning or deep learning, will help to solve the difficult problem. We hope that collaboration between the generalized evolution of copulas and AI will contribute to the study of various kinds of dependence structures transforming autonomously.

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